Self-avoiding random walks on a family of diamond-type hierarchical lattices

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We have used the exact renormalization-group method proposed by Dhar to study self-avoiding random walks (SAW's) on a family of hierarchical lattices. The generator of the lattices is made of l branches and each branch has m bonds. We expect that since the lattices are infinitely ramified, the critical exponents of SAW's should be different from that on finitely ramified lattices and belong to a new universal class. We calculated the critical exponents α, ν, and γ under the condition $l<m$, and, with D_f the fractal dimension, obtained the scaling law $D_f\nu=2-\alpha$, which agrees with other authors's conclusions. When $l \geq m$, we cannot work out the problem, and some discussion is given.

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The application of fractals in physics was first proposed by Mandelbrot [1]. Since fractal geometry is specially characterized by self-similarity, while Euclidean space is characterized by translational invariance, it has attracted great attention. Almost all statistical physics problems have been studied again on fractals [2–4], and one of them is self-avoiding random walks (SAW's).

SAW's were originally proposed as a model of polymer chains [5], and its connection with other models was found later [6,7]. SAW's on fractals have been studied by several authors [4,8–12]. Their research shows that SAW's on fractals are much different from those on translationally invariant lattices. It is certain that critical exponents of SAW's depend on the geometry properties of their support. On translationally invariant lattices, the Flory formula [13] describes the relation of the critical exponent ν and geometry parameters, while on fractals there may not exist a general formula. People have discussed the influences of fractal dimension, spectral dimension, and other geometry parameters on the SAW critical exponents [4,8–12]. However, we have not yet seen work on the influence of ramification.

In this paper we discussed SAW's on a family of diamond-type hierarchical lattices [14], as shown in Fig. 1, which are infinitely ramified fractals. Many physical problems about the lattices [14–17] have been discussed. The fractal generator is determined by two parameter m and l, where l denotes the number of branches and m the number of bonds in each branch. The fractal dimension is determined by the formula

$$D_f=\ln(ml)/\ln(m) . \tag{1}$$

In order to study SAW's, three functions are defined:

$$C(x)=\lim_{N\to\infty}(1/N)\sum_{n=1}^{\infty}C_n(N)x^n , \tag{2}$$

$$P(x)=\lim_{N\to\infty}(1/N)\sum_{n=2}^{\infty}P_n(N)x^n , \tag{3}$$

$$L(x)=\lim_{N\to\infty}(1/N)\sum_{n=1}^{\infty}\langle R_n^2\rangle C_n(N)x^n/C(x) , \tag{4}$$

where x is a weight factor associated with each step of the walks, $C_n(N)$ is the number of distinct n-step SAW's on a lattice with N sites, $P_n(N)$ is the number of distinct SAW loops of n steps, and $\langle R_n^2\rangle$ is the mean-square end-to-end distance for n-step SAW's.

For very large n, we get the asymptotic behavior of the above functions as x tends to $1/\mu$ from below [7]:

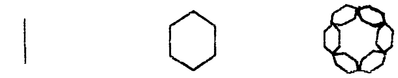
$$C(x)\approx K_1(1-x\mu)^{-\gamma}+(\text{less singular terms}) , \tag{5}$$

$$P(x)\approx K_2(1-x\mu)^{2-\alpha}+(\text{less singular terms}) , \tag{6}$$

$$L(x)\approx K_3(1-x\mu)^{-\nu}+(\text{less singular terms}) , \tag{7}$$

where α, ν, and γ are critical exponents; μ is a connective constant; and K_1, K_2, and K_3 are constants.

In the case of a hierarchical lattice, Dhar [18] has proposed a method to get the exact solution of SAW problems. The method depends on the character of the hierarchical model. Its structure at stage r is constructed by stage $r-1$ via an iteration procedure, and we can calculate generating functions stage by stage in terms of the



r = 1 r = 2 r = 3

FIG. 1. Growth of the diamond-type hierarchical lattice ($m=3$, $l=2$). The first three stages are shown.

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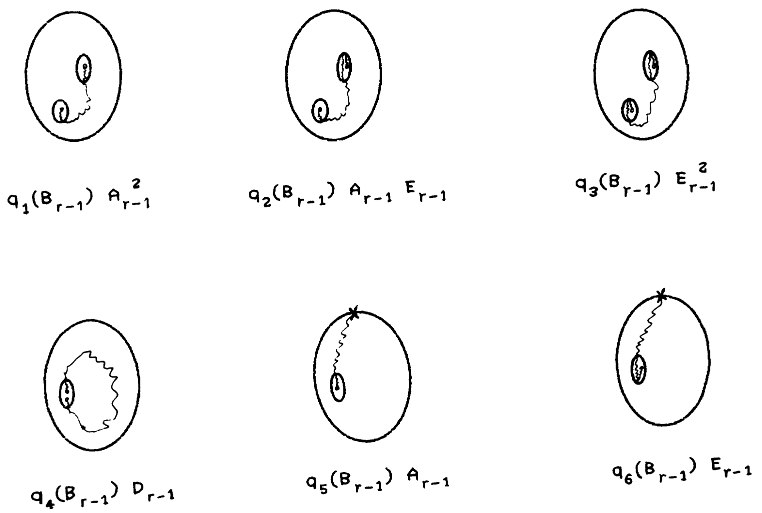

FIG. 4. The six possible rth-order SAW paths within an rth stage diamond-type lattice are shown. "×" denotes the vertex point; "○" denotes the $(r-1)$th stage diamond-type lattice; and the larger circle one denotes the rth stage.

$\ln(m)$, $\nu=1$, and $\gamma=\ln(\lambda_2^2/ml)/\ln(m)$, and got the relation $D_f\nu=2-\alpha$, which agrees with the conclusions of other authors. In Table I we have shown specific results for α, ν, and γ, together with the fractal dimension, the eigenvalue λ_2, and the connective constant μ.

We notice that the critical exponent ν is always 1 and is equal to the results of SAW's on nonloop structures. It can be understood as follows: On the infinite lattice the number of going-far-away SAW's is much larger than that of those going back along the loops, so we obtained the same critical exponent ν as that on a nonloop structure.

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APPENDIX A: THE DERIVATION OF EQ. (8) IN SEC. II

In the following we show the construction of the recursion relation (8). As an example, we consider the case $m=3$, $l=2$. All possible ways of constructing A_{r+1} are shown in Fig. 3. By summing all contributions, we get

$$A_{r+1}=2(1+B_r+B_r^2)A_r+2(1+B_r)E_r\ . \tag{A1}$$

Then for general m and l, we can similarly get

$$A_{r+1}=f_1(B_r)A_r+f_2(B_r)E_r\ , \tag{A2}$$

where

$$f_1=l(1+B_r+\cdots+B_r^{m-1})\ ,$$

$$f_2=l(1+B_r+\cdots+B_r^{m-2})\ .$$

This is just Eq. (8) in Sec. II.

APPENDIX B: THE DERIVATION OF $C(x)$

In the following we express $C(x)$ by the four restricted partition functions. In Fig. 4, we have shown all possible rth-order SAW paths within an rth stage diamond-type lattice, where $q_i(B_{r-1})$ is a polynomial in B_{r-1}. Then, summing all the contributions shown in Fig. 4 and using the same approximation as in Eq. (24), we get

$$\begin{aligned}C(x)\approx \sum_{r=2}^{\infty}\frac{1}{a(ml)^r}[&q_1(B_{r-1})A_{r-1}^2+q_2(B_{r-1})A_{r-1}E_{r-1}\\&+q_3(B_{r-1})E_{r-1}^2+q_4(B_{r-1})D_{r-1}\\&+q_5(B_{r-1})A_{r-1}+q_6(B_{r-1})E_{r-1}]\ .\end{aligned} \tag{B1}$$

This is just Eq. (30).

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